Exercise 20

Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

$$y'' - 2y' - 3y = x + 2$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' - 3y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2e^{rx}$$

Plug these formulas into equation (1).

$$r^2e^{rx} - 2(re^{rx}) - 3(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r - 3 = 0$$

Solve for r.

$$(r-3)(r+1) = 0$$

$$r = \{-1, 3\}$$

Two solutions to the ODE are e^{-x} and e^{3x}. By the principle of superposition, then,

$$y_c(x) = C_1e^{-x} + C_2e^{3x}.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 2y_p' - 3y_p = x + 2 \tag{2}$$

Part (a)

Since the inhomogeneous term is a polynomial of degree 1, the particular solution is $y_p = Ax + B$.

$$y_p = Ax + B \quad \rightarrow \quad y_p' = A \quad \rightarrow \quad y_p'' = 0$$

Substitute these formulas into equation (2).

$$(0) - 2(A) - 3(Ax + B) = x + 2$$

$$(-2A - 3B) + (-3A)x = 2 + x$$

Match the coefficients on both sides to get a system of equations for A and B.

$$\left.\begin{aligned} -2A - 3B &= 2 \\ -3A &= 1 \end{aligned}\right\}$$

Solving it yields

$$A = -\frac{1}{3} \quad \text{and} \quad B = -\frac{4}{9},$$

which means the particular solution is

$$y_p = -\frac{1}{3}x - \frac{4}{9}.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1e^{-x} + C_2e^{3x} - \frac{1}{3}x - \frac{4}{9}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Part (b)

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_1(x)e^{-x} + C_2(x)e^{3x}$$

Differentiate it with respect to x.

$$y_p' = C_1'(x)e^{-x} + C_2'(x)e^{3x} - C_1(x)e^{-x} + 3C_2(x)e^{3x}$$

If we set

$$C_1'(x)e^{-x} + C_2'(x)e^{3x} = 0, \tag{3}$$

then

$$y_p' = -C_1(x)e^{-x} + 3C_2(x)e^{3x}.$$

Differentiate it with respect to x once more.

$$y_p'' = -C_1'(x)e^{-x} + 3C_2'(x)e^{3x} + C_1(x)e^{-x} + 9C_2(x)e^{3x}$$

Substitute these formulas into equation (2).

$$\left[-C_1'(x)e^{-x} + 3C_2'(x)e^{3x} + \cancel{C_1(x)e^{-x}} + \cancel{9C_2(x)e^{3x}}\right] - 2\left[-\cancel{C_1(x)e^{-x}} + \cancel{3C_2(x)e^{3x}}\right] - 3\left[\cancel{C_1(x)e^{-x}} + \cancel{C_2(x)e^{3x}}\right] = x + 2$$

Simplify the result.

$$-C_1'(x)e^{-x} + 3C_2'(x)e^{3x} = x + 2 \tag{4}$$

Add the respective sides of equations (3) and (4) to eliminate $C_1'(x)$.

$$4C_2'(x)e^{3x} = x + 2$$

Solve for $C_2'(x)$.

$$C_2'(x) = \frac{x}{4}e^{-3x} + \frac{1}{2}e^{-3x}$$

Integrate this result to get $C_2(x)$, setting the integration constant to zero.

$$\begin{aligned} C_2(x) &= \int^x C_2'(w)\,dw \\ &= \int^x \left(\frac{w}{4}e^{-3w} + \frac{1}{2}e^{-3w}\right) dw \\ &= \frac{1}{4}\int^x we^{-3w}\,dw + \frac{1}{2}\int^x e^{-3w}\,dw \\ &= \frac{1}{4}e^{-3w}\left(-\frac{1}{9} - \frac{w}{3}\right)\bigg|^x - \frac{1}{6}e^{-3w}\bigg|^x \\ &= \frac{1}{4}e^{-3x}\left(-\frac{1}{9} - \frac{x}{3}\right) - \frac{1}{6}e^{-3x} \\ &= -\frac{1}{36}e^{-3x}(7+3x) \end{aligned}$$

Multiply both sides of equation (3) by -3, and multiply both sides of equation (4) by 1.

$$-3C_1'(x)e^{-x} - 3C_2'(x)e^{3x} = 0$$

$$-C_1'(x)e^{-x} + 3C_2'(x)e^{3x} = x + 2$$

Add the respective sides of these equations to eliminate $C_2'(x)$.

$$-4C_1'(x)e^{-x} = x + 2$$

Solve for $C_1'(x)$.

$$C_1'(x) = -\frac{x}{4}e^{x} - \frac{1}{2}e^{x}$$

Integrate this result to get $C_1(x)$, setting the integration constant to zero.

$$\begin{aligned} C_1(x) &= \int^x C_1'(w)\,dw \\ &= \int^x \left(-\frac{w}{4}e^{w} - \frac{1}{2}e^{w}\right) dw \\ &= -\frac{1}{4}\int^x we^{w} - \frac{1}{2}\int^x e^{w}\,dw \\ &= -\frac{1}{4}e^{w}(-1+w)\bigg|^x - \frac{1}{2}e^{w}\bigg|^x \\ &= -\frac{1}{4}e^{x}(-1+x) - \frac{1}{2}e^{x} \\ &= -\frac{1}{4}e^{x}(1+x) \end{aligned}$$

Therefore,

$$\begin{aligned} y_p &= C_1(x)e^{-x} + C_2(x)e^{3x} \\ &= \left[-\frac{1}{4}e^{x}(1+x)\right]e^{-x} + \left[-\frac{1}{36}e^{-3x}(7+3x)\right]e^{3x} \\ &= -\frac{1}{4}(1+x) - \frac{1}{36}(7+3x) \\ &= -\frac{x}{3} - \frac{4}{9}, \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1e^{-x} + C_2e^{3x} - \frac{1}{3}x - \frac{4}{9}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.